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Vacuum String Field Theory with B-field

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Abstract: We continue the analysis of Vacuum String Field Theory in the presence of a constant B-field. In particular we give a proof of the ratio of brane tensions is the expected one. On the wake of the recent literature we introduce wedge-like states and orthogonal projections. Finally we show a few examples of the smoothing out effects of the B-field on some of the singularities that appear in VSFT.

Keywords: Bosonic Strings, D-branes, Non-Commutative Geometry.

Contents

1. Introduction

Witten's String Field Theory (SFT), [1] in the presence of a constant background B-field has been studied in refs. [2, 3] in the field theory limit (see also the final considerations of [4]). In this limit the SFT $*$ product factorizes into the ordinary Witten $*$ product and the Moyal product. More generally, it was proven by [5, 6] that, when a B-field is switched on, the kinetic term of the SFT action remains unchanged while the three string vertex changes, being multiplied by a (cyclically invariant) non-commutative phase factor (see [5, 6]). Starting from this result, in [7] we began the exploration of the effects of a B-field in a non-perturbative regime of Vacuum String Field Theory (VSFT). In particular we were able to show that exact solutions can be written down for tachyonic lumps much in the same way as one finds analogous solutions without B-field.

In this paper we continue the analysis started in [7] and show that many results previously obtained in the matter sector of VSFT without B-field (see ref. [8]–[27]) can be generalized to the VSFT with B-field. The message we would like to convey in this paper is that the introduction of a constant background B-field is not a terrible embarrassment in developing SFT. On the other hand, it may turn out to be a useful device. In VSFT there arise several singularities. We have good reasons to believe that the well-known smoothing out effects of a B-field may help in taming some of them. At the end of this paper we present an example of such beneficial effects due to the B-field.

The paper is organized as follows. In the next section we collect a series of previously obtained results, which we need in the sequel of this paper. In section 3 we present a proof that $\mathcal{R} = 1$, where $\mathcal{R}$ is the ratio introduced in [7]. This implies that the ratio of tensions between the D25-brane and the D23-dimensional tachyonic lump is as conjectured in [7].

In section 4 we show that we can extend without much pain to the case of non-vanishing B-field a series of results which were previously derived in the case $B = 0$. In section 5 we analyze the effect of the B-field on some of the VSFT singularities. We prove that the presence of a B-field removes the singularity of the tachyonic lumps. For instance we find the GMS solitons, [27], and we show that the singular geometry of the sliver, found in [4], is smoothed out by the B-field in the lump solutions. In particular we show that the string midpoint is not confined anymore to the hyperplane of vanishing transverse dimensions.

2. Summary of previous results

Vacuum String Field Theory (VSFT) was defined in [12]. It is conjectured to represent SFT on the stable vacuum of Witten's SFT. The VSFT action has the same form as Witten's SFT action with the BRST operator Q replaced by a new one, usually denoted $\mathcal{Q}$, which has the characteristics of being universal. As a matter of fact in [14], see also [15, 16, 17, 19, 20, 21, 23, 29, 30], an explicit representation of $\mathcal{Q}$ has been proposed, purely in terms of ghost fields. Now, the equation of motion of VSFT is

$$\mathcal{Q}\Psi = -\Psi * \Psi\,. \tag{2.1}$$

One looks for non-perturbative solutions in the form

$$\Psi = \Psi_m \otimes \Psi_g\,, \tag{2.2}$$

where Ψ_g and Ψ_m depend purely on ghost and matter degrees of freedom, respectively. Then eq. (2.1) splits into

$$\mathcal{Q}\Psi_g = -\Psi_g * \Psi_g\,, \tag{2.3}$$

$$\Psi_m = \Psi_m * \Psi_m\,. \tag{2.4}$$

Equation (2.3) will not be involved in our analysis since ghosts are unaffected by the presence of a B-field. Therefore we will concentrate on solutions of (2.4).

The value of the action for such solutions is given by

$$\mathcal{S}(\Psi) = \mathcal{K}\langle\Psi_m|\Psi_m\rangle\,, \tag{2.5}$$

where $\mathcal{K}$ contains the ghost contribution. As shown in [14], $\mathcal{K}$ is infinite unless it is suitably regularized. Nevertheless, as argued in [14] a coupled solution of (2.3) and (2.4), even if it is naively singular, is nevertheless a representative of the corresponding class of smooth solutions.

In [7] we found solutions of eq. (2.4) when a constant B-field is turned on along some space directions. We consider here only the simplest B-field configuration, i.e. when B is non vanishing in the two space directions, say the 24-th and 25-th ones (see [7] for generalizations). Let us denote these directions with the Lorentz indices α and β. Then, as is well-known [28], in these two direction we have a new effective metric $G_{\alpha\beta}$, the open string metric, as well as an effective antisymmetric parameter $\theta_{\alpha\beta}$. If we set

$$B_{\alpha\beta} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \tag{2.6}$$

they take the explicit form

$$G_{\alpha\beta} = \sqrt{\mathrm{Det\,G}}\,\delta_{\alpha\beta}\,, \qquad \theta^{\alpha\beta} = -(2\pi\alpha')^2 B\epsilon^{\alpha\beta}\,, \qquad \mathrm{Det\,G} = \left(1+(2\pi B)^2\right)^2\,, \tag{2.7}$$

where $\epsilon^{\alpha\beta}$ is the 2×2 antisymmetric symbol with $\epsilon^1{}_2 = 1$ (notice the slight change of conventions with respect to [7]).

The presence of the B-field modifies the three-string vertex only in the 24-th and 25-th direction, which, in view of the D-brane interpretation, we call the transverse ones. After turning on the B-field the three-string vertex becomes

$$|V_3\rangle' = |V_{3,\perp}\rangle' \otimes |V_{3,||}\rangle \tag{2.8}$$

$|V_{3,||}\rangle$ is the same as in the ordinary case (without B-field), while

$$|V_{3,\perp}\rangle' = K_2\, e^{-E'}|\tilde{0}\rangle_{123} \tag{2.9}$$

with

$$K_2 = \frac{\sqrt{2\pi b^3}}{A^2(4a^2+3)}(\mathrm{Det}\,G)^{1/4}\,, \tag{2.10}$$

$$E' = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0} a_M^{(r)\alpha\dagger}\mathcal{V}^{rs}_{\alpha\beta,MN}a_N^{(s)\beta\dagger}\,. \tag{2.11}$$

We have introduced the indices $M=\{0,m\}, N=\{0,n\}$ and the vacuum $|\tilde{0}\rangle = |0\rangle\otimes|\Omega_{b,\theta}\rangle$, where $|\Omega_{b,\theta}\rangle$ is the vacuum with respect to the oscillators

$$a_0^{(r)\alpha} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} - i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}\,, \qquad a_0^{(r)\alpha\dagger} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} + i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}\,, \tag{2.12}$$

where $\hat{p}^{(r)\alpha}, \hat{x}^{(r)\alpha}$ are the zero momentum and position operator of the r-th string; i.e. $a_0^\alpha|\Omega_{b,\theta}\rangle = 0$. It is understood that $p^{(r)\alpha} = G^{\alpha\beta}p_\beta^{(r)}$, and

$$\left[a_M^{(r)\alpha}, a_N^{(s)\beta\dagger}\right] = G^{\alpha\beta}\delta_{MN}\delta^{rs}\,. \tag{2.13}$$

The coefficients $\mathcal{V}_{MN}^{\alpha\beta,rs}$ are given by

$$\mathcal{V}_{00}^{\alpha\beta,rs} = G^{\alpha\beta}\delta^{rs} - \frac{2A^{-1}b}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs} - ia\epsilon^{\alpha\beta}\chi^{rs}\right)\,, \tag{2.14}$$

$$\mathcal{V}_{0n}^{\alpha\beta,rs} = \frac{2A^{-1}\sqrt{b}}{4a^2+3}\sum_{t=1}^{3}\left(G^{\alpha\beta}\phi^{rt} - ia\epsilon^{\alpha\beta}\chi^{rt}\right)V_{0n}^{ts}\,, \tag{2.15}$$

$$\mathcal{V}_{mn}^{\alpha\beta,rs} = G^{\alpha\beta}V_{mn}^{rs} - \frac{2A^{-1}}{4a^2+3}\sum_{t,v=1}^{3}V_{m0}^{rv}\left(G^{\alpha\beta}\phi^{vt} - ia\epsilon^{\alpha\beta}\chi^{vt}\right)V_{0n}^{ts}\,. \tag{2.16}$$

Here, by definition, $V_{0n}^{rs} = V_{n0}^{sr}$, and

$$\phi^{rs} = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}\,, \qquad \chi^{rs} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\,. \tag{2.17}$$

These two matrices satisfy the algebra

$$\chi^2 = -2\phi\,, \qquad \phi\chi = \chi\phi = \frac{3}{2}\chi\,, \qquad \phi^2 = \frac{3}{2}\phi\,. \tag{2.18}$$

Moreover, in (2.16), we have introduced the notation

$$A = V_{00} + \frac{b}{2}\,, \qquad a = -\frac{\pi^2}{A}\,B\,, \tag{2.19}$$

Next we introduce the twist matrix C' by $C'_{MN} = (-1)^M \delta_{MN}$ and define

$$\mathcal{X}^{rs} \equiv C'\mathcal{V}^{rs}\,, \qquad r,s = 1,2\,, \qquad \mathcal{X}^{11} \equiv \mathcal{X}\,. \tag{2.20}$$

These matrices commute

$$[\mathcal{X}^{rs}, \mathcal{X}^{r's'}] = 0\,. \tag{2.21}$$

Moreover we have the following properties, which mark a difference with the $B = 0$ case,

$$C'\mathcal{V}^{rs} = \tilde{\mathcal{V}}^{sr}C'\,, \qquad C'\mathcal{X}^{rs} = \tilde{\mathcal{X}}^{sr}C'\,, \tag{2.22}$$

where tilde denotes transposition with respect to the α, β indices. Finally one can prove that

$$\begin{aligned} \mathcal{X} + \mathcal{X}^{12} + \mathcal{X}^{21} &= \mathbb{I}\,, \\ \mathcal{X}^{12}\mathcal{X}^{21} &= \mathcal{X}^2 - \mathcal{X}\,, \\ (\mathcal{X}^{12})^2 + (\mathcal{X}^{21})^2 &= \mathbb{I} - \mathcal{X}^2\,, \\ (\mathcal{X}^{12})^3 + (\mathcal{X}^{21})^3 &= 2\mathcal{X}^3 - 3\mathcal{X}^2 + \mathbb{I}\,. \end{aligned} \tag{2.23}$$

In the matrix products of these identities, as well as throughout the paper, the indices α, β must be understood in alternating up/down position: $\mathcal{X}^{\alpha}{}_{\beta}$. For instance, in (2.23) $\mathbb{I}$ stands for $\delta^{\alpha}{}_{\beta}\,\delta_{MN}$.

The lump solution we found in [7] satisfies $|S\rangle = |S\rangle * |S\rangle$ and can be written as

$$\begin{aligned} |S\rangle \;=\; & \left\{\mathrm{Det}(1-X)^{1/2}\mathrm{Det}\,(1+T)^{1/2}\right\}^{24} \exp\left(-\frac{1}{2}\eta_{\bar\mu\bar\nu}\sum_{m,n\geq 1} a_m^{\bar\mu\dagger}S_{mn}a_n^{\bar\nu\dagger}\right)|0\rangle \otimes \\ & \otimes \frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}\,G)^{1/4}}\left(\mathrm{Det}\,(\mathbb{I}-\mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I}+\mathcal{T})^{1/2}\right)\times \\ & \times \exp\left(-\frac{1}{2}\sum_{M,N\geq 0} a_M^{\alpha\dagger}\mathcal{S}_{\alpha\beta,MN}a_N^{\beta\dagger}\right)|\tilde{0}\rangle\,, \end{aligned} \tag{2.24}$$

The quantities in the first line are defined in ref. [13] with $\bar\mu, \bar\nu = 0, \ldots, 23$ denoting the parallel directions to the lump and the matrix $\mathcal{S} = C'\mathcal{T}$ is given by

$$\mathcal{T} = \frac{1}{2\mathcal{X}}\left(\mathbb{I} + \mathcal{X} - \sqrt{(\mathbb{I}+3\mathcal{X})(\mathbb{I}-\mathcal{X})}\right). \tag{2.25}$$

This is a solution to the equation

$$\mathcal{X}\mathcal{T}^2 - (\mathbb{I} + \mathcal{X})\mathcal{T} + \mathcal{X} = 0\,. \tag{2.26}$$

The solution (2.24) is interpreted as a D23-brane. Let $\mathfrak{e}$ denote the energy per unit volume, which coincides with the brane tension when $B = 0$. Then one can compute the ratio of the D23-brane energy density $\mathfrak{e}_{23}$ to the D25-brane energy density $\mathfrak{e}_{25}$ (from [12]):

$$\frac{\mathfrak{e}_{23}}{\mathfrak{e}_{25}} = \frac{(2\pi)^2}{(\mathrm{Det}\, G)^{1/4}} \cdot \mathcal{R}\,, \tag{2.27}$$

$$\mathcal{R} = \frac{A^4(3+4a^2)^2}{2\pi b^3 (\mathrm{Det}\, G)^{1/4}} \frac{\mathrm{Det}\,(\mathbb{I} - \mathcal{X})^{3/4} \mathrm{Det}\,(\mathbb{I} + 3\mathcal{X})^{1/4}}{\mathrm{Det}\,(1 - X)^{3/2} \mathrm{Det}\,(1+3X)^{1/2}}\,, \tag{2.28}$$

where $X \equiv X^{11} = CV^{11}$ is the matrix called M in [13]. X is the matrix relevant to the sliver solution in VSFT. The 1 in the denominator of (2.28) stands for δ_{nm}.

It was conjectured in [7] that $\mathcal{R}$ equals 1 so that the ratio (2.27) is exactly what is expected for the ratio of a flat static D25-brane energy density and a D23-brane energy density in the presence of the B-field (2.6). This is indeed so as we will prove in the next section.

3. Proof that $\mathcal{R} = 1$

This section is devoted to the proof of

$$\mathcal{R} = 1\,. \tag{3.1}$$

What we need is compute the ratio of $\mathrm{Det}\,(\mathbb{I} - \mathcal{X})$ and $\mathrm{Det}\,(\mathbb{I} + 3\mathcal{X})$ with respect to the squares of $\mathrm{Det}\,(1 - X)$ and $\mathrm{Det}\,(1 + 3X)$, respectively. To this end we follow the lines of ref. [24]. To start with we rewrite $\mathcal{V}^{11} \equiv \mathcal{V}$ in a more convenient form. Following [24], we introduce the vector notation $|v_e\rangle$ and $|v_0\rangle$ by means of

$$|v_e\rangle_n = \frac{1 + (-1)^n}{2} \frac{A_n}{\sqrt{n}}\,, \qquad |v_o\rangle_n = \frac{1 - (-1)^n}{2} \frac{A_n}{\sqrt{n}}\,, \tag{3.2}$$

The constants A_n are as in [8]. Now we can write

$$\begin{aligned}
\mathcal{V}_{00} &= \left(1 - \frac{2A^{-1}b}{4a^2+3}\right) \mathbf{1}\,, \\
\mathcal{V}_{0n} &= -\frac{2A^{-1}\sqrt{2b}}{4a^2+3} \mathbf{1}\, \langle v_e|_n + i\sqrt{\frac{2b}{3}} \frac{4aA^{-1}}{4a^2+3} \mathbf{e}\, \langle v_o|_n\,, \qquad \mathcal{V}_{0n} = (-1)^n \mathcal{V}_{n0}\,, \\
\mathcal{V}_{nm} &= \left(V_{nm} - \frac{4A^{-1}}{4a^2+3} (|v_e\rangle\langle v_e| + |v_o\rangle\langle v_o|)_{nm}\right) \mathbf{1} + i\frac{8}{\sqrt{3}} \frac{aA^{-1}}{4a^2+3} (|v_e\rangle\langle v_o| - |v_o\rangle\langle v_e|)_{nm}\, \mathbf{e}\,,
\end{aligned} \tag{3.3}$$

where we have understood the indices α, β. They can be reinserted using

$$\mathbf{1}^\alpha{}_\beta = \delta^\alpha{}_\beta\,, \qquad \mathbf{e}^\alpha{}_\beta = \epsilon^\alpha{}_\beta \tag{3.4}$$

Now $\mathcal{X} = C'\mathcal{V}$ can be written in the following block matrix form

$$\mathcal{X} = \begin{pmatrix} (1-2Kb)\mathbf{1} & -2K\sqrt{2b}\,\mathbf{1}\,\langle v_e| + 4iaK\sqrt{\frac{2b}{3}}\,\mathbf{e}\,\langle v_o| \\ -2K\sqrt{2b}|v_e\rangle\,\mathbf{1} & X\mathbf{1} - 4K\,\mathbf{1}\,(|v_e\rangle\langle v_e| - |v_o\rangle\langle v_o|) \\ +4iaK\sqrt{\frac{2b}{3}}|v_o\rangle\,\mathbf{e} & +\frac{8}{\sqrt{3}}iaK\,\mathbf{e}\,(|v_e\rangle\langle v_o| + |v_o\rangle\langle v_e|) \end{pmatrix}, \tag{3.5}$$

where all m, n as well as all α, β indices are understood, $K = \frac{A^{-1}}{4a^2+3}$.

The first determinant we have to compute is the one of the matrix $\mathbb{I} - \mathcal{X}$. Using (3.5) we extract from $\mathbb{I} - \mathcal{X}$ the factor $2bK$ and represent the rest in the block form

$$\frac{1}{2bK}(\mathbb{I} - \mathcal{X}) = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{3.6}$$

By a standard formula, the determinant of the r.h.s. is given by the determinant of $\mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B}$. After some algebra and using the obvious identity $\langle v_o|v_e\rangle = 0$, one gets

$$\begin{aligned} \mathcal{D} - \mathcal{C}\mathcal{A}^{-1}\mathcal{B} &= \begin{pmatrix} 1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o| & 0 \\ 0 & 1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o| \end{pmatrix} \\ &= \left(1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o|\right)\mathbf{1}\,. \end{aligned}$$

The rest of the computation is straightforward,

$$\begin{aligned} \mathrm{Det}(\mathbb{I} - \mathcal{X}) &= (2bK)^2\left(\mathrm{Det}\left(1 - X - \frac{4}{3}A^{-1}|v_o\rangle\langle v_o|\right)\right)^2 \\ &= (2bK)^2\,(\mathrm{Det}(1-X))^2\left(\mathrm{Det}\left(1 - \frac{4}{3}A^{-1}\,\frac{1}{1-X}\,|v_o\rangle\langle v_o|\right)\right)^2 \\ &= \left(\frac{b}{A}\right)^4\left(\frac{1}{4a^2+3}\right)^2(\mathrm{Det}(1-X))^2\,. \end{aligned} \tag{3.7}$$

In the last step we have used the identities, see [24],

$$\mathrm{Det}\left(1 - \frac{4}{3}A^{-1}\,\frac{1}{1-X}\,|v_o\rangle\langle v_o|\right) = 1 - \frac{4}{3}A^{-1}\,\langle v_o|\frac{1}{1-X}\,|v_o\rangle \tag{3.8}$$

and

$$\langle v_o|\frac{1}{1-X}\,|v_o\rangle = \frac{3}{4}V_{00}\,. \tag{3.9}$$

The treatment of $\mathrm{Det}\,(\mathbb{I} + 3\mathcal{X})$ is less trivial. We start again by writing $(\mathbb{I} + 3\mathcal{X})$ in block matrix form

$$\mathbb{I} + 3\mathcal{X} = \begin{pmatrix} (4-6Kb)\mathbf{1} & -6K\sqrt{2b}\,\mathbf{1}\,\langle v_e| + 4iaK\sqrt{6b}\,\mathbf{e}\,\langle v_o| \\ -6K\sqrt{2b}\,|v_e\rangle\,\mathbf{1} & (1+3X)\mathbf{1} - 12K\,\mathbf{1}\,(|v_e\rangle\langle v_e| - |v_o\rangle\langle v_o|) \\ +4iaK\sqrt{6b}\,|v_o\rangle\,\mathbf{e} & +8\sqrt{3}iaK\,\mathbf{e}\,(|v_e\rangle\langle v_o| + |v_o\rangle\langle v_e|) \end{pmatrix} \tag{3.10}$$

and set

$$\mathbb{I} + 3\mathcal{X} \equiv (4 - 6bK)\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{3.11}$$

Therefore

$$\begin{aligned}\mathrm{Det}(\mathbb{I}+3\mathcal{X}) &= (4-6bK)^2\mathrm{Det}\left(\mathcal{D}-\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right)\\ &= (4-6bK)^2\,\left(\mathrm{Det}\,(1+3X)\right)^2\,\mathrm{Det}\left(\frac{1}{1+3X}\left(\mathcal{D}-\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right)\right). \end{aligned} \tag{3.12}$$

The last expression is formal. In fact X has an eigenvalue $-1/3$ which renders the r.h.s. of (3.12) ill-defined. To avoid this we follow [24] and introduce the regularized inverse

$$Y_\varepsilon = \frac{1}{1+3X-\varepsilon^2 X}\,, \tag{3.13}$$

where ε is a small parameter, and replace it into (3.12). After some algebra we find

$$Y_\varepsilon\,\left(\mathcal{D}-\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right) = \mathcal{A}\cdot\mathcal{B}\,. \tag{3.14}$$

The matrices in the r.h.s. are given by

$$\mathcal{A} = \begin{pmatrix} 1+\alpha Y_\varepsilon|v_e\rangle\langle v_e| + \beta Y_\varepsilon|v_o\rangle\langle v_o| & 0\\ 0 & 1+\alpha Y_\varepsilon|v_e\rangle\langle v_e| + \beta Y_\varepsilon|v_o\rangle\langle v_o| \end{pmatrix}, \tag{3.15}$$

where

$$\alpha = -\frac{24K}{2-3bK}\,, \qquad \beta = 12K\frac{2-A^{-1}}{2-3bK}\,, \tag{3.16}$$

and

$$\mathcal{B} = \begin{pmatrix} 1 & \lambda Y_\varepsilon|v_2\rangle\langle v_o| + \mu Y_\varepsilon|v_o\rangle\langle v_e|\\ -\lambda Y_\varepsilon|v_e\rangle\langle v_o| - \mu Y_\varepsilon|v_o\rangle\langle v_e| & 1 \end{pmatrix}, \tag{3.17}$$

where,

$$\lambda = \frac{\gamma}{1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle}\,, \qquad \mu = \frac{\gamma}{1+\beta\langle v_0|Y_\varepsilon|v_0\rangle}\,, \qquad \gamma^2+\alpha\beta = -\frac{4}{V_{00}}\beta\,. \tag{3.18}$$

Now, after some computation,

$$\mathrm{Det}\,\mathcal{A} = \left(1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle\right)^2\,\left(1+\beta\langle v_o|Y_\varepsilon|v_o\rangle\right)^2 \tag{3.19}$$

and

$$\mathrm{Det}\,\mathcal{B} = \left(1+\frac{\gamma^2\langle v_e|Y_\varepsilon|v_e\rangle\langle v_o|Y_\varepsilon|v_0\rangle}{\left(1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle\right)\,\left(1+\beta\langle v_o|Y_\varepsilon|v_o\rangle\right)}\right)^2. \tag{3.20}$$

As a consequence

$$\mathrm{Det}\,\mathcal{A}\,\mathrm{Det}\,\mathcal{B} = \left(1+\alpha\langle v_e|Y_\varepsilon|v_e\rangle + \beta\langle v_o|Y_\varepsilon|v_0\rangle\left(1-\frac{4}{V_{00}}\langle v_e|Y_\varepsilon|v_e\rangle\right)\right)^2. \tag{3.21}$$

Now we can remove the regulator ε by using the basic result of [24]:

$$\lim_{\varepsilon\to 0}\left(1-\frac{4}{V_{00}}\langle v_e|Y_\varepsilon|v_e\rangle\right)\,\langle v_o|Y_\varepsilon|v_o\rangle = \frac{\pi^2}{12V_{00}} \tag{3.22}$$

and

$$\langle v_e|\frac{1}{1+3X}|v_e\rangle = \frac{V_{00}}{4}\,. \tag{3.23}$$

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Inserting this result in (3.21) we find

$$\mathrm{Det}\,\mathcal{A}\,\mathrm{Det}\,\mathcal{B} = \frac{A^2}{(8a^2A + 6V_{00})^2}\left(8a^2 + \frac{2\pi^2}{A^2}\right)^2 . \tag{3.24}$$

As a consequence of eqs. (3.12), (3.14), (3.21) and (3.24) we find

$$\frac{\mathrm{Det}(\mathbb{I}+3\mathcal{X})}{(\mathrm{Det}(1+3X))^2} = \frac{4}{(4a^2+3)^2}\left(8a^2 + \frac{2\pi^2}{A^2}\right)^2 . \tag{3.25}$$

Finally, substituting this and (3.7) into $\mathcal{R}$, we get

$$\mathcal{R} = \frac{A^4(3+4a^2)^2}{2\pi b^3(\det G)^{1/4}}\frac{\mathrm{Det}(\mathbb{I}-\mathcal{X})^{3/4}\mathrm{Det}(\mathbb{I}+3\mathcal{X})^{1/4}}{\det(1-X)^{3/2}\mathrm{Det}(1+3X)^{1/2}} = 1\,. \tag{3.26}$$

This is what we wanted to show. It implies

$$\frac{\mathfrak{e}_{23}}{\mathfrak{e}_{25}} = \frac{(2\pi)^2}{(\det G)^{1/4}} \tag{3.27}$$

which corresponds to the expected result for this ratio, as explained in [7]. We remark that (3.25) implies that the eigenvalue $-1/3$ is also contained in the spectrum of $\mathcal{X}$ with double multiplicity with respect to X.

4. Some results in VSFT with B-field

In this section we deal with a couple of results which are natural extensions of analogous results with $B = 0$.

4.1 Wedge-like states

Wedge states were introduced in [26]. They are geometrical states in that they can be defined simply by means of a conformal map of the unit disk to a portion of it. They are spanned by an integer n: the limit for $n \to \infty$ is the sliver Ξ, which is interpreted as the D25-brane. Wedge states also admit a representation in terms of oscillators $a_n^\dagger$ with $n > 0$,

$$|W_n\rangle = \mathcal{N}_n^{26}\, e^{-\frac{1}{2}a^\dagger C T_n a^\dagger}|0\rangle \tag{4.1}$$

which is specified by the matrix T_n, $n > 1$. It can be shown that, see [26], T_n satisfy a recursive relation which can be solved in terms of the matrix T characterizing the sliver state ($T = CS$, S being the sliver matrix). The normalization $\mathcal{N}$ can also be derived from a recursion relation. Since all these results are essentially based on equations which can be generalized to the case when a B-field is present and were in fact derived in [7], it is easy to deduce that analogous results will hold also when a B-field is turned on.

The generalized wedge states will be the tensor product of a factor like (4.1) for the the 24 directions in which the components of the B-field are zero and

$$|\mathcal{W}_n\rangle = \mathcal{N}_n^2\, e^{-\frac{1}{2}a^\dagger C' \mathcal{T}_n a^\dagger}|\tilde{0}\rangle \tag{4.2}$$

for the other two directions. From now on we will be concerned with the determination of $\mathcal{T}_n$ and $\mathcal{N}_n$. We start from the hypothesis that

$$[\mathcal{X}^{rs}, \mathcal{T}_n] = 0\,, \qquad C'\mathcal{T}_n = \tilde{\mathcal{T}}_n C' \tag{4.3}$$

whose consistency we will verify a posteriori.

Now we define $\mathcal{T}_2 = 0$ and the sequence of states

$$|\mathcal{W}_{n+1}\rangle = |\mathcal{W}_n\rangle * |\mathcal{W}_2\rangle\,. \tag{4.4}$$

Using [7, eqs. (4.4) and (4.7)], with $\Sigma = \begin{pmatrix} C'\tilde{\mathcal{T}}_n & 0 \\ 0 & 0 \end{pmatrix}$, we find the recursion relation

$$\begin{aligned}\mathcal{T}_{n+1} &= \mathcal{X}^{11} + (\mathcal{X}^{12}, \mathcal{X}^{21}) \left(1 - \begin{pmatrix} \mathcal{T}_n\mathcal{X}^{11} & \mathcal{T}_n\mathcal{X}^{12} \\ 0 & 0 \end{pmatrix}\right)^{-1} \begin{pmatrix} \mathcal{T}_n\mathcal{X}^{21} \\ 0 \end{pmatrix} \\ &= \mathcal{X}\,\frac{1-\mathcal{T}_n}{1-\mathcal{T}_n\mathcal{X}}\,,\end{aligned} \tag{4.5}$$

where use has been made of the second equation in (2.23). Solving this recursion relation, [26], we can write

$$\mathcal{T}_n = \frac{\mathcal{T} + (-\mathcal{T})^{n-1}}{1-(-\mathcal{T})^n}\,. \tag{4.6}$$

Notice that this sequence of states can be extended to $|\mathcal{W}_1\rangle$ defined by $\mathcal{T}_1 = 1$. An analogous recursion relation applies also to the normalization factors. Once solved, it gives

$$\mathcal{N}_n = K_2^{-1}\mathrm{Det}\left(\frac{1-\mathcal{T}^2}{1-(-\mathcal{T})^{n+1}}\right)^{1/2} \tag{4.7}$$

The constant K_2 is defined in [7, eq. (2.19)]. The relations (4.3) are now easy to verify.

The limit of $\mathcal{T}_n$ as $n \to \infty$ is $\mathcal{T}$ (i.e. the deformation of the lump), provided $\lim \mathcal{T}^n = 0$. In turn, the latter holds if the eigenvalues of $\mathcal{T}$ are in absolute value less then 1, as those of T are. This is very likely in view of the results on the ratio of determinants in the last section.[1]

4.2 Orthogonal projectors with B-field

In the presence of a background B-field it is also possible to construct other projectors than the one shown in (2.24). To show this we follow ref. [14]. The treatment is very close to sections 3 and 5 of that reference, and the main purpose of this subsection is to stress some differences with it. As usual we will be concerned only with the transverse part of the projectors, the parallel being exactly the same as in [14], and will denote the transverse part of the solution (2.24) by $|\mathcal{S}_\perp\rangle$.

We start by introducing the projection operators

$$\rho_1 = \frac{1}{(\mathbb{I}+\mathcal{T})(\mathbb{I}-\mathcal{X})}\left[\mathcal{X}^{12}(\mathbb{I}-\mathcal{T}\mathcal{X}) + \mathcal{T}(\mathcal{X}^{21})^2\right], \tag{4.8}$$

$$\rho_2 = \frac{1}{(\mathbb{I}+\mathcal{T})(\mathbb{I}-\mathcal{X})}\left[\mathcal{X}^{21}(\mathbb{I}-\mathcal{T}\mathcal{X}) + \mathcal{T}(\mathcal{X}^{12})^2\right]. \tag{4.9}$$

[1] An analysis of the eigenvalues of $\mathcal{X}$ has been recently announced in [31].

They satisfy

$$\rho_1^2 = \rho_1 \,, \qquad \rho_2^2 = \rho_2 \,, \qquad \rho_1 + \rho_2 = \mathbb{I} \tag{4.10}$$

i.e. they project onto orthogonal subspaces. Moreover, if we use the superscript ^{T} to denote transposition with respect to the indices N, M and α, β, we have

$$\rho_1^T = \tilde{\rho}_1 = C'\rho_2 C' \,, \qquad \rho_2^T = \tilde{\rho}_2 = C'\rho_1 C' \,. \tag{4.11}$$

Now, in order to find another solution of the equation $|\Psi\rangle * |\Psi\rangle = |\Psi\rangle$, distinct from $|\mathcal{S}_\perp\rangle$, we make the following ansatz:

$$|\mathcal{P}_\perp\rangle = \left(-\xi\tau a^\dagger \; \zeta \cdot a^\dagger + \kappa\right) |\mathcal{S}_\perp\rangle \,, \tag{4.12}$$

where $\xi = \{\xi_N^\alpha\}$, $\zeta = C'\xi$ and τ is the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acting on the indices α and β. κ is a constant to be determined and ξ is required to satisfy the constraints:

$$\rho_1 \xi = 0 \,, \qquad \rho_2 \xi = \xi \,, \qquad \text{i.e. } \tilde{\rho}_1 \zeta = \zeta \,, \qquad \tilde{\rho}_2 \zeta = 0 \,. \tag{4.13}$$

Using (4.10) and (4.13) it is simple to prove that

$$\zeta^T f(\mathcal{X}^{rs}, \mathcal{T})\, \xi = 0 \,, \qquad \xi^T f(\tilde{\mathcal{X}}^{rs}, \tilde{\mathcal{T}}) \zeta = 0 \tag{4.14}$$

for any function f. Now, imposing $|\mathcal{P}_\perp\rangle * |\mathcal{S}_\perp\rangle = 0$ we determine κ:

$$\kappa = -\frac{1}{2} \zeta^T \tau \left(\mathcal{V}\mathcal{K}^{-1}\right)_{11} \xi - \frac{1}{2} \xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{11} \tau \zeta \,, \tag{4.15}$$

where

$$\mathcal{K} = \mathbb{I} - \mathcal{T}\mathcal{X} \,, \qquad \mathcal{V} = \begin{pmatrix} \mathcal{V}^{11} & \mathcal{V}^{12} \\ \mathcal{V}^{21} & \mathcal{V}^{22} \end{pmatrix} . \tag{4.16}$$

Next we compute $|\mathcal{P}_\perp\rangle * |\mathcal{P}_\perp\rangle$. This gives

$$|\mathcal{P}_\perp\rangle * |\mathcal{P}_\perp\rangle = \frac{1}{2} \left(\xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{12} \tau\zeta + \zeta^T \tau \left(\mathcal{V}\mathcal{K}^{-1}\right)_{21} \xi\right) \left(-a^\dagger \tau \xi \; a^\dagger \cdot \zeta + \kappa\right) |\mathcal{S}_\perp\rangle \,, \tag{4.17}$$

where use has been made of the identities

$$\begin{aligned}
\zeta^T \tau \left(\mathcal{V}\mathcal{K}^{-1}\right)_{11} \xi &= \zeta^T \tau \left(\mathcal{V}\mathcal{K}^{-1}\right)_{22} \xi = -\zeta^T \tau \left(\mathcal{V}\mathcal{K}^{-1}\right)_{12} \xi = \xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \xi \,, \\
\xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{11} \tau\zeta &= \xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{22} \tau\zeta = -\xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{21} \tau\zeta = \zeta^T \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \tau\zeta \,, \\
\xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{12} \tau\zeta &= \zeta^T \frac{1}{\mathbb{I} - \mathcal{T}^2} \tau\zeta \,, \qquad \zeta^T \tau \left(\mathcal{V}\mathcal{K}^{-1}\right)_{21} \xi = \xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2} \xi \,.
\end{aligned} \tag{4.18}$$

Similarly one can prove that

$$\zeta^T \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \tau\zeta = \xi^T \tau \frac{\mathcal{T}}{\mathbb{I} - \mathcal{T}^2} \xi \,, \qquad \zeta^T \frac{1}{\mathbb{I} - \mathcal{T}^2} \tau\zeta = \xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2} \xi \,. \tag{4.19}$$

So, in order for $|\mathcal{P}_\perp\rangle$ to be a projector, we have to impose

$$\left(\xi^T \left(\mathcal{V}\mathcal{K}^{-1}\right)_{12} \tau\zeta + \zeta^T \tau \left(\mathcal{V}\mathcal{K}^{-1}\right)_{21} \xi\right) = 2\, \xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2} \xi = 2 \,. \tag{4.20}$$

Using this and following [14], it is simple to prove that

$$\langle \mathcal{P}_\perp | \mathcal{P}_\perp \rangle = \left(\zeta^T \frac{1}{\mathbb{I} - \mathcal{T}^2} \tau \zeta \ \xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2} \xi \right) \langle \mathcal{S}_\perp | \mathcal{S}_\perp \rangle = \langle \mathcal{S}_\perp | \mathcal{S}_\perp \rangle \tag{4.21}$$

thanks to (4.19) and (4.20).

Therefore, under the condition

$$\xi^T \tau \frac{1}{\mathbb{I} - \mathcal{T}^2} \xi = 1 \tag{4.22}$$

the BPZ norm of $|\mathcal{P}_\perp\rangle + |\mathcal{S}_\perp\rangle$ is twice the norm of $|\mathcal{S}_\perp\rangle$. As a consequence the sum of these two states, once they are tensored by the corresponding 24-dimensional complements defined in [14], represent a couple of parallel D23-branes.

Similarly one can construct more complicated brane configurations as suggested in [14].

5. Some effects of the B-field

So far we have seen that many results of VSFT without B-field are accompanied by parallel results in the presence of the B-field. In this section we would like to show that this is not merely a formal replication, but that in some cases a non-vanishing background B-field affects in a significant way the form of such results. Precisely we would like to show that a B-field has the effect of smoothing out some of the singularities that appear in the VSFT.

5.1 Low energy limit

In [4] it was shown that the geometry of the lower-dimensional lump states representing Dp-branes is singular. This can be seen both in the zero slope limit $\alpha' \to 0$ and as an exact result. It can be briefly stated by saying that the midpoint of the string is confined on the hyperplane of vanishing transverse coordinates. It is therefore interesting to see whether the presence of a B-field modifies this situation. Moreover it is also well known that soliton solutions of field theories defined on a non-commutative space describe Dp-branes ([27, 32]). It is then interesting to see if we can recover at least the simplest GMS soliton, using the particular low energy limit, i.e. the limit of [28], that gives a non-commutative field theory from a string theory with a B-field turned on.

To discuss this limit we first reintroduce the closed string metric $g_{\alpha\beta}$ as $g\,\delta_{\alpha\beta}$. Now we take $\alpha' B \gg g$, in such a way that G, θ and B are kept fixed. The limit is described by means of a parameter ϵ going to 0 as in [4] ($\alpha' \sim \epsilon$). We could also choose to parametrize the $\alpha' B \gg g$ condition by sending B to infinity, keeping g and α' fixed and operating a rescaling of the string modes as in [3], of course at the end we get identical results. By looking at the exponential of the 3-string field theory vertex in the presence of a B-field

$$\sum_{r,s=1}^{3} \left(\frac{1}{2} \sum_{m,n \geq 1} G_{\alpha\beta} a_m^{(r)\alpha\dagger} V_{mn}^{rs} a_n^{(s)\beta\dagger} + \sqrt{\alpha'} \sum_{n \geq 1} G_{\alpha\beta} p_{(r)}^{\alpha} V_{0n}^{rs} a_n^{(s)\beta\dagger} + \right.$$
$$\left. + \alpha' \frac{1}{2} G_{\alpha\beta} p_{(r)}^{\alpha} V_{00}^{rs} p_{(s)}^{\beta} + \frac{i}{2} \sum_{r<s} p_{\alpha}^{(r)} \theta^{\alpha\beta} p_{\beta}^{(s)} \right) \tag{5.1}$$

we see that the limit is characterized by the rescalings

$$\begin{aligned} V_{mn} &\longrightarrow V_{mn}\,, \\ V_{m0} &\longrightarrow \sqrt{\epsilon} V_{m0}\,, \\ V_{00} &\longrightarrow \epsilon V_{00} \end{aligned} \tag{5.2}$$

$G_{\alpha\beta}$ and $\theta^{\alpha\beta}$ are kept fixed. Their explicit dependence on g, α' and B will be reintroduced at the end of our calculations in the form

$$G_{\alpha\beta} = \frac{(2\pi\alpha' B)^2}{g}\delta_{\alpha\beta}\,, \qquad \theta = \frac{1}{B}\,. \tag{5.3}$$

Substituting the leading behaviors of V_{MN} in eqs. (2.16), and keeping in mind that $A = V_{00} + \frac{b}{2}$, the coefficients $\mathcal{V}^{\alpha\beta,rs}_{MN}$ become

$$\begin{aligned} \mathcal{V}^{\alpha\beta,rs}_{00} &\longrightarrow G^{\alpha\beta}\delta^{rs} - \frac{4}{4a^2+3}\left(G^{\alpha\beta}\phi^{rs} - ia\hat{\epsilon}^{\alpha\beta}\chi^{rs}\right), \\ \mathcal{V}^{\alpha\beta,rs}_{0n} &\longrightarrow 0\,, \\ \mathcal{V}^{\alpha\beta,rs}_{mn} &\longrightarrow G^{\alpha\beta} V^{rs}_{mn}\,. \end{aligned} \tag{5.4}$$

We see that the squeezed state (2.24) factorizes in two parts: the coefficients $\mathcal{V}^{\alpha\beta,11}_{mn}$ reconstruct the full 25-dimensional sliver, while the coefficients $\mathcal{V}^{\alpha\beta,11}_{00}$ take a very simple form

$$\mathcal{S}^{\alpha\beta}_{00} = \frac{2|a|-1}{2|a|+1} G^{\alpha\beta} \equiv s\, G^{\alpha\beta}\,. \tag{5.5}$$

The soliton lump with this choice of the coefficients $\mathcal{V}^{\alpha\beta,rs}_{MN}$ will be called $|\hat{\mathcal{S}}\rangle$

$$\begin{aligned} |\hat{\mathcal{S}}\rangle = &\left\{\mathrm{Det}(1-X)^{1/2}\mathrm{Det}\,(1+T)^{1/2}\right\}^{24} \exp\left(-\frac{1}{2}\eta_{\bar\mu\bar\nu}\sum_{m,n\geq 1} a^{\bar\mu\dagger}_m S_{mn} a^{\bar\nu\dagger}_n\right)|0\rangle \otimes \\ &\otimes \exp\left(-\frac{1}{2}G_{\alpha\beta}\sum_{m,n\geq 1} a^{\alpha\dagger}_m S_{mn} a^{\beta\dagger}_n\right)|0\rangle \frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\det G)^{1/4}} \times \\ &\times \left(\mathrm{Det}(\mathbb{I}-\mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I}+\mathcal{T})^{1/2}\right)\exp\left(-\frac{1}{2}s a^{\alpha\dagger}_0 G_{\alpha\beta} a^{\beta\dagger}_0\right)|\Omega_{b,\theta}\rangle\,, \end{aligned} \tag{5.6}$$

where $\bar\mu, \bar\nu = 0, \dots, 23$ and $\alpha, \beta = 24, 25$. In the low energy limit we have also

$$\mathrm{Det}(\mathbb{I}-\mathcal{X})^{1/2}\mathrm{Det}(\mathbb{I}+\mathcal{T})^{1/2} = \frac{4}{4a^2+3}\,\det(1-X)\,\frac{4a}{2a+1}\,\det(1+T)\,. \tag{5.7}$$

So the complete lump state becomes

$$\begin{aligned} |\hat{\mathcal{S}}\rangle = &\left\{\mathrm{Det}(1-X)^{1/2}\mathrm{Det}\,(1+T)^{1/2}\right\}^{26} \exp\left(-\frac{1}{2}G_{\mu\nu}\sum_{m,n\geq 1} a^{\mu\dagger}_m S_{mn} a^{\nu\dagger}_n\right)|0\rangle \otimes \\ &\otimes \frac{4a}{2a+1}\,\frac{b^2}{\sqrt{2\pi b^3}(\mathrm{Det}\,G)^{1/4}}\,\exp\left(-\frac{1}{2}s a^{\alpha\dagger}_0 G_{\alpha\beta} a^{\beta\dagger}_0\right)|\Omega_{b,\theta}\rangle\,, \end{aligned} \tag{5.8}$$

where $\mu, \nu = 0, \ldots, 25$ and $G_{\mu\nu} = \eta_{\bar{\mu}\bar{\nu}} \oplus G_{\alpha\beta}$. The first line of (5.8) is the usual 25-dimensional sliver up to a simple rescaling of $a_n^{\alpha\dagger}$. The norm of the lump is now regularized by the presence of a which is directly proportional to B: $a = -\frac{\pi^2}{A} B$. Using

$$|x\rangle = \sqrt{\frac{2\sqrt{\operatorname{Det} G}}{b\pi}} \exp\left[-\frac{1}{b} x^\alpha G_{\alpha\beta} x^\beta - \frac{2}{\sqrt{b}} i a_0^{\alpha\dagger} G_{\alpha\beta} x^\beta + \frac{1}{2} a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right] |\Omega_{b,\theta}\rangle \tag{5.9}$$

we can calculate the projection onto the basis of position eigenstates of the transverse part of the lump state

$$\begin{aligned} \langle x| e^{-\frac{s}{2}(a_0^\dagger)^2} |\Omega_{b,\theta}\rangle &= \sqrt{\frac{2\sqrt{\operatorname{Det} G}}{b\pi}} \frac{1}{1+s} e^{-\frac{1-s}{1+s}\frac{1}{b} x^\alpha x^\beta G_{\alpha\beta}} \\ &= \sqrt{\frac{2\sqrt{\operatorname{Det} G}}{b\pi}} \frac{1}{1+s} e^{-\frac{1}{2|a|b} x^\alpha x^\beta G_{\alpha\beta}} . \end{aligned} \tag{5.10}$$

The transverse part of the lump state in the x representation is then

$$\langle x|\hat{\mathbb{S}}_\perp\rangle = \frac{1}{\pi} e^{-\frac{1}{2|a|b} x^\alpha x^\beta G_{\alpha\beta}} . \tag{5.11}$$

Using the form (5.3) of $G_{\alpha\beta}$ and $\theta^{\alpha\beta}$ and the expression for a where we explicitate the dependence on g and α', [7]

$$a = \frac{\theta}{4A}\sqrt{\operatorname{Det} G} = -\frac{2\pi^2(\alpha')^2 B}{b\, g} \tag{5.12}$$

we obtain the simplest soliton solution of [27] (see also [32] and references therein):

$$e^{-\frac{1}{2|a|b} x^\alpha x^\beta G_{\alpha\beta}} \longrightarrow e^{-\frac{x^\alpha x^\beta \delta_{\alpha\beta}}{|\theta|}} \tag{5.13}$$

which corresponds to the $|0\rangle\langle 0|$ projector in the harmonic oscillator Hilbert space of [27]. We notice that the profile and the normalization of $\langle x|\hat{\mathbb{S}}_\perp\rangle$ do not depend on b.

As compared to [4], the B-field provides a natural realization of the regulator for the tachyonic soliton introduced ad hoc there. This beneficial effect of the B-field is confirmed by the fact that the projector (5.8) is no longer annihilated by x_0

$$\begin{aligned} x_0 \exp\left(-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right) |\Omega_{b,\theta}\rangle &= i\,\frac{\sqrt{b}}{2}\,(a_0 - a_0^\dagger) \exp\left(-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right) |\Omega_{b,\theta}\rangle \\ &= -i\,\frac{\sqrt{b}}{2}\left[\frac{4a}{2a+1}\right] a_0^\dagger \exp\left(-\frac{1}{2} s a_0^{\alpha\dagger} G_{\alpha\beta} a_0^{\beta\dagger}\right) |\Omega_{b,\theta}\rangle . \end{aligned}$$

Therefore, at least in the low energy limit, the singular structure found in [4] has disappeared.

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5.2 The string midpoint

In the previous subsection a very interesting question has been raised. It concerns the string midpoint. It was shown in [4] that, in the absence of a B-field, the string midpoint in the lower dimensional lumps is confined to the hyperplane (D-brane) of vanishing transverse coordinates. We have seen above that this is not anymore the case when a constant background B-field is present, at least in the field theory limit. One might deduce from this that the string midpoint is not confined anymore in the full VSFT either. However such conclusion is far from self-evident. As we will see in the sequel, the field theory limit (tachyon) contribution to the string midpoint is only one out of an infinite set of other contributions which characterize the full theory and it is not inconceivable that there might be a cancellation between the field theory limit contribution and all the other terms. Evaluating the exact string midpoint position in the full VSFT is in fact a non-trivial and interesting problem. We intend to address it in this subsection.

The oscillator expansion for the transverse string coordinates is, [5], setting $\alpha' = 1/2$,

$$x^\alpha(\sigma) = x_0^\alpha + \frac{\theta^{\alpha\beta}}{\pi} p_{0,\beta}\left(\sigma - \frac{\pi}{2}\right) + \sqrt{2}\sum_{n=1}^{\infty}\left[x_n^\alpha \cos(n\sigma) + \frac{\theta^{\alpha\beta}}{\pi}\frac{1}{n} p_{n,\beta}\sin(n\sigma)\right]. \qquad (5.14)$$

Therefore the string midpoint is specified by

$$x^\alpha\left(\frac{\pi}{2}\right) = x_0^\alpha + \sqrt{2}\sum_{n=1}^{\infty}(-1)^n\left[x_{2n}^\alpha - \frac{\theta^{\alpha\beta}}{\pi}\frac{1}{2n-1} p_{2n-1,\beta}\right]. \qquad (5.15)$$

It is more convenient to pass to the operator basis $a_N^\alpha, a_N^{\alpha\dagger}$, which satisfies the algebra (2.13) and are related to x_n, p_n by

$$x_n^\alpha = \frac{i}{\sqrt{2n}}\left(a_n^\alpha - a_n^{\alpha\dagger}\right), \qquad p_{n,\alpha} = \sqrt{\frac{n}{2}} G_{\alpha\beta}\left(a_n^\beta + a_n^{\beta\dagger}\right), \qquad (5.16)$$

while the analogous relation for x_0, p_0 is given by eq. (2.12) with the specification that throughout this section, for simplicity, we fix $b = 2$.

Now, confinement of the string midpoint means

$$x^\alpha\left(\frac{\pi}{2}\right)|\mathbb{S}_\perp\rangle = 0\,. \qquad (5.17)$$

Evaluating the l.h.s. we get

$$\begin{aligned} x^\alpha\left(\frac{\pi}{2}\right)|\mathbb{S}_\perp\rangle &= -\frac{i}{\sqrt{2}}(a^\dagger + a^\dagger\mathbb{S})_0^\alpha|\mathbb{S}_\perp\rangle - i\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n}}(a^\dagger + a^\dagger\mathbb{S})_{2n}^\alpha|\mathbb{S}_\perp\rangle - \\ &- \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n-1}}\frac{\theta^{\alpha\beta}}{\pi}G_{\beta\gamma}(a^\dagger - a^\dagger\mathbb{S})_{2n-1}^\gamma|\mathbb{S}_\perp\rangle\,. \end{aligned} \qquad (5.18)$$

Confinement requires that this vanish. In order to write this condition in compact form, we introduce the 2×2-matrix-valued vector

$$\Theta = |\nu\rangle\,\mathbf{1} + |\mu\rangle\,\mathbf{e}\,, \qquad (5.19)$$

where

$$|\nu\rangle = \{\nu_0, \nu_{2n}\}\,, \qquad \nu_0 = \frac{1}{\sqrt{2}}\,, \qquad \nu_{2n} = \frac{(-1)^n}{\sqrt{2n}}\,,$$
$$|\mu\rangle = \{\mu_{2n-1}\}\,, \qquad \mu_{2n-1} = i\pi\, B\,\frac{(-1)^n}{\sqrt{2n-1}}\,. \tag{5.20}$$

Now the confinement condition for the string midpoint can be written as

$$\mathcal{S}C'\,\Theta = -\Theta\,, \qquad \text{or, equivalently}\,, \qquad \tilde{\mathcal{T}}\,\Theta = -\Theta\,, \qquad \text{i.e. } \mathcal{T}\tilde{\Theta} = -\tilde{\Theta}\,. \tag{5.21}$$

Due to (2.25) an eigenvalue -1 of $\mathcal{T}$ corresponds to an eigenvalue $-1/3$ of $\mathcal{X}$ with the same eigenvector. Let us rewrite $\mathbb{I} + 3\mathcal{X}$ as

$$\mathbb{I} + 3\mathcal{X} = \mathcal{Y}\,\mathbf{1} + \mathcal{Z}\,\mathbf{e}\,. \tag{5.22}$$

Then eq. (5.21) becomes $(\mathbb{I} + 3\mathcal{X})\,\tilde{\Theta} = 0$, which in turn corresponds to the two equations

$$\mathcal{Y}\,|\nu\rangle + \mathcal{Z}\,|\mu\rangle = 0\,, \tag{5.23}$$
$$\mathcal{Z}\,|\nu\rangle - \mathcal{Y}\,|\mu\rangle = 0\,. \tag{5.24}$$

It is useful to further split $\mathcal{Y}$ as $\mathcal{Y} = \mathcal{Y}_0 + \mathcal{Y}_1$, where $\mathcal{Y}_0 = \mathcal{Y}(B=0)$. Using (3.10) one obtains

$$\mathcal{Y}_0 = \begin{pmatrix} 4(1 - A^{-1}) & -4A^{-1}\langle v_e| \\ & \\ -4A^{-1}|v_e\rangle & 1 + 3X - 4\,A^{-1}(|v_e\rangle\langle v_e| - |v_o\rangle\langle v_o|) \end{pmatrix}, \tag{5.25}$$

$$\mathcal{Y}_1 = 12\,H \begin{pmatrix} 1 & \langle v_e| \\ & \\ |v_e\rangle & |v_e\rangle\langle v_e| - |v_o\rangle\langle v_o| \end{pmatrix}, \tag{5.26}$$

$$\mathcal{Z} = 8\sqrt{3}\,i\,a\,K \begin{pmatrix} 0 & \langle v_o| \\ & \\ |v_o\rangle & |v_e\rangle\langle v_o| + |v_o\rangle\langle v_e| \end{pmatrix}, \tag{5.27}$$

where $H = \frac{4}{3}\frac{a^2 A^{-1}}{4a^2+3}$.

Now let us express the previous equations in a more explicit form. To this end we conform to the notation of section 2 and write

$$|\nu\rangle = \nu_0 \oplus |\nu_e\rangle\,,$$
$$|\mu\rangle = -i\pi B|\lambda_o\rangle\,,$$

where

$$|\nu_e\rangle_n = \frac{1 + (-1)^n}{2}\nu_n\,, \qquad \nu_n = \frac{(-1)^{n/2}}{\sqrt{n}}\,, \tag{5.28}$$

$$|\lambda_o\rangle_n = \frac{1 - (-1)^n}{2}\lambda_n\,, \qquad \lambda_n = \frac{(-1)^{(n+1)/2}}{\sqrt{n}}\,. \tag{5.29}$$

We remark that $|\nu\rangle$ is the eigenvector corresponding to the eigenvalue $-1/3$ of $\mathcal{X}(B=0)$, introduced in [4]; and that $|\lambda_o\rangle$ is the eigenvector with eigenvalue $-1/3$ of X, introduced in [16]. As a consequence one has

$$\mathcal{Y}_0\,|\nu\rangle = 0\,, \qquad (1+3X)|\lambda_o\rangle = 0\,. \tag{5.30}$$

The first equation can be rewritten as

$$\langle v_e|\nu_e\rangle = V_{00}\nu_0\,, \tag{5.31}$$

$$(1+3X)|\nu_e\rangle = 4\nu_0|v_e\rangle\,. \tag{5.32}$$

Remarkably enough, all the other equations from (5.23) and (5.24), after using (5.31) and the second equation in (5.30), reduce to a single one

$$\langle v_o|\lambda_o\rangle = \sqrt{\frac{2}{3}}\pi\,. \tag{5.33}$$

Therefore, since eqs. (5.30) have been proved independently, confinement of the string midpoint holds or not according to whether eq. (5.33) is true or not. Now, the l.h.s. of this equation is

$$\langle v_o|\lambda_o\rangle = \sum_{n\ \mathrm{odd}} (-1)^{(n+1)/2}\frac{A_n}{n}\,. \tag{5.34}$$

The latter series can be summed with standard methods and gives

$$\langle v_o|\lambda_o\rangle = \frac{9-2\sqrt{3}\pi}{6}\,. \tag{5.35}$$

Therefore (5.33) is definitely not satisfied. So we can conclude that the string midpoint in the presence of a B-field is *not confined* on the hyperplane that identifies the D23-brane.

6. Conclusion

In this paper we have shown that the introduction of a B-field in VSFT does not prevent us from obtaining parallel results to those obtained when $B=0$. Once the formalism is set up, the formal complications brought about by the B-field are far from scaring. The calculations of section 3 and 4 are examples of this fact. On the other hand a non-vanishing background B-field may have advantageous aspects. The smoothing out effects of B on the UV divergences of non-commutative field theories are well-known. The aim of this paper was to start exploring the effects of a B-field on the diverse singularities that appear in VSFT. We have verified that the singular geometry of the lump solutions, pointed out in [4], disappears in the presence of a B-field, in particular the string midpoint is not confined any longer to stay on the D-brane.

We remark that this *deconfinement* might mean also that the left-right factorization characteristic of the sliver solution, [14, 29, 30], is not possible for lump solutions with B-field. However it looks like there are other aspects of VSFT which may be fruitfully extended to VSFT with B-field. For instance, the series of wedge-like states introduced

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in section 4.1 seem to suggest that the geometric nature of the wedge states, [11], persists also in the presence of a B-field. This is confirmed by the results obtained recently in [33], where the presence of a B-field has been dealt with entirely geometrically. It would be interesting to know whether, for instance, the analogs of butterfly states in a constant B background, [34, 35, 36], can be constructed. Another interesting subject concerns the connection, if any, of our approach with ref. [37, 38].

Note added. This paper appeared on the net simultaneously with [39, 40], which partially overlap with it.

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